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Quantization of monotonic twist maps

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Abstract. Using an approach suggested by Moser, classical Hamiltonians are generated that provide an interpolating flow to the stroboscopic motion of maps with a monotonic twist condition. The quantum properties of these Hamiltonians are then studied in analogy with recent work on the semiclassical quantization of systems based on Poincaré surfaces of section. For the generalized standard map, the correspondence with the usual classical and quantum results is shown, and the advantages of the quantum Moser Hamiltonian demonstrated. The same approach is then applied to the free motion of a particle on a 2-torus, and to the circle billiard. A natural quantization condition based on the eigenphases of the unitary time-development operator is applied, leaving the exact eigenvalues of the torus, but only the semiclassical eigenvalues for the billiard; an explanation for this failure is proposed. It is also seen how iterating the classical map commutes with the quantization.

1. Introduction

Quantization techniques based on a Poincaré surface of section (hereafter PSS) have provided much inspiration recently in the study of the semiclassical properties of chaotic systems. The semiclassical quantization procedure of Bogomolny [2] involved the T-operator as the semiclassical version of the classical area-preserving map of the chosen PSS onto itself. Likewise the scattering approach of Doron and Smilansky [3, 4], used the semiclassical S-matrix as the analogue of the Poincaré scattering map [5], with the PSS being the billiard boundary. This PSS was also important in [6] where the corrections to the leading-order semiclassical results were found for billiard systems.

Interest in the area-preserving maps themselves goes back further, with the standard [7–9], baker [10, 11] and cat maps [12, 13] all providing much insight into the correspondence between chaotic classical dynamics and quantum kinematics; see also [14, 15]. In such cases viewing the mapping as the 'stroboscopic' picture of the flow generated by an underlying Hamiltonian is not always necessary or even possible. The quantum properties of interest are derivable from the unitary time-development operator for one period, and the details associated with times in between the applications of this operator have up to now been of secondary importance.

Nevertheless, it is still of interest to know whether an interpolating flow can be found for a given map. An affirmative response to this question for monotonic twist maps was given in [1], a brief review of which will be given in section 2. The construction of the flow is, loosely speaking, the inverse of the Poincaré construction where a map is derived from a flow. But it is also highly non-unique, since for generic maps there are many possible interpolating flows. To aid calculation, the simplest possible flow (linear in time and space) is chosen, and in section 3, the classical mechanics that results from this is presented for

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generalized standard maps, the free motion of a particle on a 2-torus and for the boundary map of the circle billiard.

In section 4 this approach is quantized, and the consequences of this are investigated and compared with known quantum results. This is the central purpose of this paper. With the non-uniqueness of the classical prescription, and the generic appearance of gauge-like fields that lead to operator ordering problems, the quantized Moser Hamiltonian touches on some of the deeper aspects of the quantum–classical correspondence. So, although it is something of a mathematical curiosity as far as semiclassical mechanics is concerned, Moser's beautiful idea provides a new and interesting setting for studying the semiclassical properties of area-preserving maps.

2. Monotonic twist maps and Moser's Hamiltonian

Consider a two-dimensional phase space labelled by (x, p) such that points (x, p) and $(x + 1, p)$ are identified. Such a cylindrical phase space was adopted in [1] though the periodicity condition can be relaxed without affecting the results. An area-preserving map of this phase space onto itself, $x_1 = f(x_0, p_0)$, $p_1 = g(x_0, p_0)$, is then a monotonic twist map if

$$\frac{\partial x_1}{\partial p_0} > 0 \tag{1}$$

so that as the initial point moves up the cylinder, it maps ('twists') increasingly to the right. The alternative inequality sign (<) can also be used; the most important point that is required here is that it has a constant sign. One of the important results in [1] was that (1) can be directly related to the Legendre condition $\partial^2 L/\partial \dot{x}^2 > 0$ on the Lagrangian that Moser introduces to provide the interpolating motion between (x_0, p_0) and (x_1, p_1). This in turn translates into the interpolating Hamiltonian being *optical*, $\partial^2 H/\partial p^2 > 0$, the mathematical consequences of which have been investigated by Bialy and Polterovich [16], who have also generalized Moser's result to higher dimensional phase spaces [17].

Any area-preserving map can be expressed in terms of a generating function, $h(x_0, x_1)$. The mapping is then expressed implicitly by

$$p_0 = -\frac{\partial h(x_0, x_1)}{\partial x_0} \tag{2}$$

$$p_1 = \frac{\partial h(x_0, x_1)}{\partial x_1}. \tag{3}$$

This is automatically area-preserving, conserves the periodicity, and the analogue to the twist condition is $\partial^2 h/\partial x_0 \partial x_1 < 0$. The search for an interpolating flow consists of finding a Lagrangian $L(x(t), \dot{x}(t), t)$ which satifies the following conditions.

- There exists a flow $x(t)$ which solves the Lagrange equation and which satisfies the boundary conditions

$$x(0) = x_0 \qquad x(1) = x_1\,. \tag{4}$$

- The generating function for the mapping, $h(x_0, x_1)$ can be written as the action along the classical path $x(t)$

$$h(x_0, x_1) = \int_0^1 \mathrm{d}t\, L(x(t), \dot{x}(t), t)\,. \tag{5}$$

If there is an underlying interpolating flow, then it could either be from a time-independent system in two-dimensions, or a periodic time-dependent system in one-dimension. In the former case, the flow would exist outside the PSS and the mapping would represent the points of intersection of the flow with the PSS. However, Moser adopted the simpler approach of looking for a flow defined purely on the PSS, which is more naturally associated with the latter case. In this way any extra dimensions 'hidden' by the mapping can be ignored.

Moser's approach provides a way of constructing the interpolating Lagrangian in terms of the map by inverting (5) under the assumption that the only extremal motion that carries x_0 to x_1 is linear in time,

$$x(t) = x_0 + t(x_1 - x_0)\,. \tag{6}$$

This is the central assumption in [1], and is important in simplifying the Euler–Lagrange equations sufficiently to allow the inversion to take place. The other, less important, assumption is related to the fact that any Lagrangian is only defined up to an arbitrary 'gauge'; the addition of a total time derivative of a function of position and time does not affect the classical mechanics. It is important to include this freedom when inverting (5), since such a gauge is generally necessary to correctly determine the value of the action between all pairs of points; see [1] for more details. Moser chooses a gauge that varies linearly in time. It will be shown below that any other time dependence of the gauge does not change the classical or quantum results.

The Lagrangian that gives (5) subject to (4) can then be written in terms of the map $h(x_0, x_1)$ as [1]

$$L(x, y = \dot{x}, t) = -\int_0^y \mathrm{d}y'(y - y')h_{xx'}\left(x - y't, x + y'(1-t)\right) + y m_x(t, x) + m_t(t, x) \tag{7}$$

where the function $m(t, x)$ was chosen in [1] with linear time dependence for simplicity, i.e.

$$m_x(t, x) = -(1-t)h_x(x, x) + t h_{x'}(x, x) \tag{8}$$

$$m_t(t, x) = h(x, x)\,. \tag{9}$$

This will be used in the following, but in general the only necessary condition on $m(t, x)$ is

$$m(1, x_1) - m(0, x_0) = h(x_0, x_1) - S_0(x_0, x_1) \tag{10}$$

for all x_0 and x_1. Here S_0 is the action corresponding to the time integral along the extremal linear path of the first term on the right of (7). It is straightforward to check that the Lagrangian given by (7) possesses the desired properties.

It is worth pointing out at this juncture that although the induced motion is free, the Lagrangian (7) need not be simply related by a gauge transformation to the 'free-particle' Lagrangian, $L = \dot{x}^2/2$. This is because any Lagrangian which is only a function of velocity, and which has $L_{\dot{x}\dot{x}} \neq 0$, gives free-particle motion, as can be seen from the Euler equations,

$$\dot{p} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L(\dot{x})}{\partial \dot{x}} = \ddot{x} L_{\dot{x}\dot{x}}(\dot{x}) = \frac{\partial L(\dot{x})}{\partial x} = 0\,. \tag{11}$$

These Lagrangians cannot generally be related to the usual free-particle Lagrangian using a gauge. This fact is important in giving Moser's approach sufficient latitude to impose free-particle motion on all monotonic twist maps.

In [1], Moser goes on to show how the discontinuities in the periodically time-continued Lagrangian can be dealt with by shifting them inside the interval $t \in [0, 1)$ and then smoothing using a convolution with a C^∞ sampling function of arbitrarily small extent in time. In this way, the mapping action can be preserved, as can the twist condition throughout the whole interval. While this is necessary to achieve a mathematically valid continuation in time, of more interest here is studying the quantum implications of Moser's Hamiltonian.

3. Some classical mechanical examples.

In order to show Moser's technique at work, three examples will now be given. The generalized standard map is a useful example of a time-dependent system where the 'PSS' is the full phase space viewed stroboscopically. Moreover, the results found can be compared against known results as a test of Moser's approach. The free motion on a torus and the circle billiard then provide cases where the PSS is smaller than the full phase space, and so Moser's imposed flow is totally artificial.

3.1. The generalized standard map

Consider first the generalized standard map generating function where the 'kick' is at the endpoint of the mapping

$$h(x_0, x_1) = \frac{(x_1 - x_0)^2}{2} - \mathrm{V}(x_1) . \tag{12}$$

This was alluded to in [1], but no classical or quantum implications were considered. Using (7) this gives the Hamiltonian

$$H_{\mathrm{Mos}}(x, p, t) = \tfrac{1}{2}(p - m_x(t, x))^2 - m_t(t, x) \tag{13}$$

for all $m(t, x)$ satisfying (10); Moser's choice gives $m(t, x) = -tV(x)$. It should be remembered that this Hamiltonian is strictly only valid on $t \in [0, 1)$.

At first sight, (13) bears no obvious relation to the usual generalized standard map Hamiltonian

$$H_{\mathrm{SM}}(x, p, t) = \frac{p^2}{2} + V(x)\delta_{\mathrm{per}}(t) \tag{14}$$

restricted to just one time step, where $\delta_{\mathrm{per}}(t)$ is the periodic delta function with spikes at the integers. Only when (13) is extended to all time does the connection appear. This can be easily done using Moser's choice of $m(t, x)$, and replacing the t in m_x by $\{t\}$, which is the fractional part of t. It is then simple to show that

$$L_{\mathrm{Mos}} = L_{\mathrm{SM}} - \frac{\mathrm{d}}{\mathrm{d}t}\big(\{t\}V(x)\big) \tag{15}$$

and so both give the same classical mechanics. In principle this equivalence should hold whatever the choice of $m(t, x)$.

3.2. Free motion on a torus

Consider a rectangle with sides of length 1 and a, and periodic boundary conditions. Free motion on this surface is equivalent to free motion on a 2-torus. Such a geometry has been studied extensively in the past, particularly when the torus has an obstacle on it such that the particle will scatter specularly. Indeed, with a circular obstacle, this system was one of the first to be proven ergodic [18] (see also [19]), while with a square obstacle, Richens and Berry [20] showed how the integrability conditions requisite for a global foliation of phase space with 2-tori are subtly broken, leaving the system pseudo-integrable.

For simplicity, the torus in this case will be taken as empty, and the PSS will be taken as a circle parallel to the 'side' of unit length. In this case the classical map is

$$p_1 = p_0 \tag{16}$$

$$x_1 = x_0 + \frac{ap_0}{\sqrt{P^2 - p_0^2}} \bmod 1 \tag{17}$$

where P is the total momentum of the particle on the torus. With the modulo condition absorbed by the identification $x \equiv x + 1$, this mapping can be represented by the generating function

$$h(x_0, x_1) = P\sqrt{a^2 + (x_1 - x_0)^2} \tag{18}$$

which satisfies the monotonic twist map condition. The corresponding Moser Lagrangian and Hamiltonian are then

$$L_{\mathrm{Mos}}(x, \dot{x}, t) = P\sqrt{a^2 + \dot{x}^2} \tag{19}$$

$$H_{\mathrm{Mos}}(x, p, t) = -a\sqrt{P^2 - p^2}. \tag{20}$$

Because the velocity on the PSS has been chosen to be constant and equal to $x_1 - x_0$, it is easy to see that any map satisfying $h(x_0, x_1) = h(x_1 - x_0)$ will beget a Moser Lagrangian of exactly the same form but with $\dot{x}$ replacing $x_1 - x_0$; this can be seen by comparing (18) with (19).

3.3. Billiard boundary maps

Much of the recent work using PSS's for quantization has considered billiard systems where the boundary forms a useful PSS [21]. Defining the scalar distances around the boundary x_0, x_1, the generating function for the mapping can be written as

$$h(x_0, x_1) = P\rho(x_0, x_1) \tag{21}$$

where P is the momentum of the particle inside the billiard, and $\rho(x_0, x_1) = |\boldsymbol{r}(x_0) - \boldsymbol{r}(x_1)|$ is the length of the chord connecting x_0 to x_1. For simplicity, only convex billiards will be assumed here to avoid problems caused by ghosts. Such maps are monotonic twist maps, though they satisfy the opposite sign of inequality to (1); the mixed derivative of $h(x_0, x_1)$ being [6]

$$\frac{\partial^2 h(x_0, x_1)}{\partial x_0 \partial x_1} = P\frac{\cos\theta(x_0, x_1)\cos\theta(x_1, x_0)}{\rho(x_0, x_1)} \tag{22}$$

where the cosines are defined by

$$\cos\theta(x, y) = \hat{\boldsymbol{n}}(x) \cdot \frac{\boldsymbol{r}(x) - \boldsymbol{r}(y)}{\rho(x, y)}.$$

and $\hat{n}(x)$ is the outward normal vector at x. For convex billiards (22) is always positive, and finite if the boundary is smooth.

Using this, and the substitution $z = x - y't$, the Moser Lagrangian can be written as

$$L_{\mathrm{Mos}}(x, y = \dot{x}, t) = -P \int_0^y \mathrm{d}y'(y - y') \frac{\cos\theta(z, z + y')\cos\theta(z + y', z)}{\rho(z, z + y')} + yP\,. \tag{23}$$

As a simple test of this approach consider the circle billiard where the circle has a radius a. In terms of the PSS coordinates introduced above, the map and its corresponding Moser Lagrangian and Hamiltonian, can be written as

$$h(x_0, x_1) = 2Pa\sin(\pi(x_1 - x_0)) \tag{24}$$

$$L_{\mathrm{Mos}}(x, \dot{x}, t) = 2Pa\sin(\pi\dot{x}) \tag{25}$$

$$H_{\mathrm{Mos}}(x, p, t) = \frac{p}{\pi}\tan^{-1}\left(\frac{\sqrt{(2\pi Pa)^2 - p^2}}{p}\right) - \frac{1}{\pi}\sqrt{(2\pi Pa)^2 - p^2}\,. \tag{26}$$

The fact that the Lagrangian and Hamiltonian are independent of x is a natural consequence of the symmetry of the circle billiard. For convex billiards with more complicated boundary shapes, it is expected that the Hamiltonian will generally depend on both x and t, where the dependence on x will preserve the periodicity of the phase space [1], and t is understood as a fictitious time. More significantly however, even for the circle billiard, there is no known classical analogue to (26) against which Moser's result can be tested.

Indeed, a general map will not have a known underlying Hamiltonian, let alone one that produces free motion across the PSS. This is particularly true for cases such as the billiard boundary map where the true motion takes place in a phase space which is larger than the PSS. Hence, H_{Mos} will have no exact classical analogue against which to compare. In the next section however, it will be seen that some comparison can be made at the quantum level by analogy with the semiclassical maps of [2] and [3].

4. Quantizing the Moser Hamiltonian

4.1. The standard map

In view of the classical equivalence seen in (15), it could now be asked what is the point of discussing the quantum mechanics since the physics behind the two will be the same. But this would be to miss some of the advantages that can be afforded by using the quantum H_{Mos}. Having a sawtooth time-dependence in the Hamiltonian is immediately less singular than having delta functions, and allows the quantum properties of the generalized standard map to be derived in a more straightforward manner. Indeed, for any choice of $m(t, x)$ with the correct boundary condition, the exact one-step wavefunctions can be found be gauging away the $m(t, x)$ in (13) when the Hamiltonian is quantized in the usual way. This leads to normalized solutions being

$$\psi_k(x, t) = \exp\left(\mathrm{i}\left(kx - \omega t + \frac{m(t, x)}{\hbar}\right)\right) \tag{27}$$

where $k = p/\hbar$ and $\omega = E/\hbar = \hbar k^2/2$. If $V(x)$ is periodic in x then this will be reflected in k being quantized to integer multiples of 2π.

In this basis, it is very easy to calculate the unitary propagator as the amplitude for going from a free-particle state with $k = 2\pi n$ to a free-particle state with $k = 2\pi n'$ in one time-step:

$$U_{n'n} = \langle n'|U|n\rangle = \sum_l \langle n'|l, 1\rangle\langle l, 0|n\rangle \tag{28}$$

where the $|l, t\rangle$ states are those of (27) with $k = 2\pi l$. This gives

$$U_{n'n} = \sum_l \mathrm{e}^{-2\mathrm{i}\hbar(n'\pi)^2} \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}x' \mathrm{e}^{2\pi\mathrm{i}((l-n')x'+(n-l)x)} \exp\left(\frac{\mathrm{i}}{\hbar}\left(m(1,x') - m(0,x)\right)\right). \tag{29}$$

From this it can be seen that only the boundary condition (10) on $m(t, x)$ is important, and hence the unitary propagator is independent of the gauge, as was hoped. For the Chirikov–Taylor standard map, the boundary condition is $m(1, x') - m(0, x) = -V(x) = -K\cos(2\pi x)$ which leaves

$$U_{n'n} = \mathrm{e}^{-2\mathrm{i}\hbar(n'\pi)^2}\mathrm{i}^{n'-n}\mathrm{J}_{n-n'}\left(\frac{K}{\hbar}\right) \tag{30}$$

where $\mathrm{J}_n(z)$ is a Bessel function [22]. This is the exact unitary propagator [23, 24].

Furthermore, this approach leads to a way of writing the unitary propagator as a time-ordered exponential [25], rather than as the product of two unitary operators as is the case in the conventional approach. Namely, U for one time step can be written as

$$U_1 = \hat{\mathrm{T}}\exp\left(-\frac{\mathrm{i}}{\hbar}\int_0^1 \mathrm{d}t\, H_{\mathrm{Mos}}(x, \hat{p}, t)\right) \tag{31}$$

where $\hat{T}$ indicates time-ordering. This can be related to the conventional unitary propagator for a 'kick' at the end

$$U_1 = \exp\left(-\mathrm{i}\frac{V(x)}{\hbar}\right)\exp\left(-\mathrm{i}\frac{p^2}{2\hbar}\right) \tag{32}$$

by using the well-known operator identity

$$\mathrm{e}^{-A}\mathrm{e}^{-C} = \hat{T}_\lambda \exp\left[-\int_0^1 \mathrm{d}\lambda \mathrm{e}^{-\lambda A}(A + C)\mathrm{e}^{\lambda A}\right] \tag{33}$$

with λ-independent operators, A and C, that need not commute†. The connection is then almost trivial if $m(t, x) = -tV(x)$ is chosen in H_{Mos}, and λ is interpreted as the time in (33). But this is a very specific choice of gauge, and hides many potential problems connected with manipulating the time-ordered integral in the exponent of (33). Indeed, it is partly coincidence that by using the most straightforward approach in Moser's formalism, the results can be shown to be exactly the same as those in the usual approach.

It is also interesting to see that in making the connection above, the operator ordering choice when quantizing Moser's Hamiltonian must be the symmetric ordering, $qp \rightarrow$

† This can be proved using $R(\mu) = \mathrm{e}^{-\mu A}\mathrm{e}^{-\mu C}$ by differentiating and then integrating $R(\mu)$ with respect to μ, and then setting $\mu = 1$.

$(qp+pq)/2$. This is because when (32) is substituted into (33) the following manipulation must be used:

$$\exp\left(-\mathrm{i}\frac{\lambda V(x)}{\hbar}\right)\hat{p}^2\exp\left(\mathrm{i}\frac{\lambda V(x)}{\hbar}\right)\equiv\left(\hat{p}+\lambda V_x(x)\right)^2.$$

The resulting quantum Hamiltonian is identical to $\hat{H}_{\mathrm{Mos}}$ only if the classical H_{Mos} is correctly symmetrized before quantization. In general, the formalism that Moser introduced, will lead to operator ordering ambiguities, and while symmetrization works for the generalized standard maps, there is no guarantee that it will work in all cases.

So far Moser's approach has not revealed anything new in the generalized standard map and, as has been seen, with the appropriate transformations, the classical and quantum Moser results could have been deduced without Moser's approach. The fact that Moser's new approach gave the known results in this case, is not so surprising because the free-motion he imposes in between mappings *is* just the exact classical mechanics of the generalized standard map. The next two examples outline the quantum implications of Moser's exact, classical approach for cases where the imposed classical flow is completely artificial.

4.2. *Free motion on the torus*

As with the simplicity of the classical mechanics of this model, so the quantum mechanics (and semiclassical mechanics) is also straightforward. Quantizing (20) in the usual way, the exact eigenstates are simply the plane waves along the PSS, $\psi_n(x)=\exp(\pm 2\pi \mathrm{i} nx)$, which satisfy the periodic boundary conditions. Hence the eigenvalues for this problem are

$$\varepsilon_n=-a\hbar\sqrt{k^2-(2\pi n)^2} \tag{34}$$

where $P=\hbar k$. It is clear that only a finite range of n gives real eigenvalues, and so $\hat{H}_{\mathrm{Mos}}$ is only Hermitian in a finite-dimensional Hilbert space which is parametrized by k. Classically this is because the energy shell acts as the boundary of the PSS, which, likewise, restricts the classical motion. Having an 'energy-dependent' Hamiltonian will thus be a generic feature of the application of the Moser approach to the PSSs of two-dimensional conservative systems.

The most important question now is how these quantum results are related to the known results for the 2-torus. Consider for a moment the same question for any conservative system, where the PSS is dimensionally smaller than phase space. In such a case the Hamiltonian for the full flow and the Moser Hamiltonian will be distinctly different. Indeed, there might seem no *a priori* reason why quantizing Moser's approach should lead to any of the quantum results (e.g. the eigenvalues) of the full system. Yet the equivalence of the stroboscopic flows on the PSS for the two approaches suggests that the quantum properties of the full system can be studied through the unitary time-development operator U_1. This is the quantum equivalent of the classical Poincaré map within Moser's approach. In analogy with other semiclassical [2] or semiquantum [3] maps of the PSS onto itself, the quantization condition on k to get the eigenvalues of the full system is then

$$\det(\mathbf{I}-\mathbf{U}_1(k))=0 \tag{35}$$

where it is assumed that the elements of $\mathbf{U}_1(k)$ implicitly contain any phases that appear through the boundary conditions imposed. These phases will be Maslov-like, with, for example, a factor of $\exp(-\mathrm{i}\pi)$ for each time a path corresponding to a given element of $\mathbf{U}_1(k)$ encounters a boundary with Dirichlet conditions imposed. For generic systems, the determination of the correct phases for each element will be a complex task. Fortunately, for the systems studied here, they are easy to include.

It might be wondered how this quantization condition can be motivated without appealing to analogy. Certainly, with no assumed knowledge as to the underlying billiard system, (35) has not the same physical significance. Once this knowledge is included, however, it is then possible to use the fact that the eigenstates of the underlying system are completely independent of the fictitious time used in the Poincaré mapping. Any quantity on the PSS associated with an eigenstate must then behave likewise.

Time development in fictitious time is achieved (stroboscopically) by acting on a given function with U_1. So any function on the PSS associated with an eigenfunction of the billiard must be completely unchanged when operated on in this way, even up to a phase. Otherwise, there would still be some fictitious time dependence that could be observed when normalizing an eigenfunction of the billiard with $\psi^*(r, t)$ and $\psi(r, t)$ corresponding to the same real time t, but different fictitious times. This requirement for complete independence of fictitious time leads to (35) as the quantization condition.

For the 2-torus, in the basis of plane waves on the PSS, U_1 is diagonal with elements $(U_1)_{nn} = \exp(i\alpha_n)$ where the α_n are the quasi-energies. For cases such as this where Moser's Hamiltonian is independent of fictitious time, they will be related to the eigenvalues of H_{Mos} by $\alpha_n = -\varepsilon_n/\hbar + \mu$ where μ is the phase arizing from the boundary conditions. It then follows from (35) that the quantization condition becomes $\alpha_n = 2m\pi$ for some n and m. Using the connection with the ε_n, and the fact that $\mu = 0$ for periodic boundary conditions this can be written

$$k^2 = (2\pi n)^2 + \left(\frac{2\pi m}{a}\right)^2 \tag{36}$$

which gives the exact quantum eigenvalues for the free particle on the torus. Note also that if Dirichlet, rather than periodic, boundary conditions are imposed on the sides of length 1, then the same problem can be treated classically by using one end of the rectangle as the PSS and doubling the the length travelled in the orthogonal direction to account for the reflection. This simply replaces a in (36) by $2a$, which again gives the exact eigenvalues.

This exactness cannot be explained away as for the standard map, since the 'correct' classical motion now extends out of the PSS, and so Moser's imposed flow is totally artificial. Indeed, in all cases where the PSS and phase space are different, Moser's flow will necessarily be artificial, and it is only the equivalence of the maps that allows any connection to be made with the quantum results. Nevertheless, the simple (almost trivial) geometry underlying the above system makes it somewhat special. It cannot therefore be implied from this that Moser's approach will always lead to the exact eigenvalues, as will be seen below.

4.3. Boundary maps for billiards

The space and time independence of H_{Mos} when applied to the circle billiard (26), also makes the quantum mechanics very simple. The exact eigenstates are the free-particle waves on the boundary, $\psi_n(x) = \exp(\pm 2\pi i n x)$, and the quasi-energies α_n can be written

$$\alpha_n = -\frac{\varepsilon_n}{\hbar} + \pi = 2\sqrt{(ka)^2 - n^2} - 2n\tan^{-1}\left(\frac{\sqrt{(ka)^2 - n^2}}{n}\right) + \pi \tag{37}$$

where the π comes from the phase change on reflection of every path crossing the PSS (boundary with Dirichlet conditions imposed) and $k = P/\hbar$ can take any value provided $ka > n$.

As for the 2-torus, the quantization condition can be expected to come from (35). This looks very similar to the billiard quantization condition in [3], with U_1 replacing S, where S is a semiquantum scattering matrix. Applying (35) leads to the allowed values of k satisfying $\alpha_n(k) = 2m\pi$, or

$$\eta_n(k) - n\tan^{-1}\left(\frac{\eta_n(k)}{n}\right) = m\pi - \frac{\pi}{2} \tag{38}$$

where $\eta_n(k) = \sqrt{(ka)^2 - n^2}$. This is almost the condition that comes from the WKB approach [6]. The difference is interesting and important; the WKB version of (38) has an extra $-\pi/4$ on the right-hand side coming from the caustic that every path encounters when travelling from one point on the boundary to another. This extra term is necessary in order to define the correct semiclassical energies. The reason that Moser's approach fails to incorporate this term is because the restriction to the PSS loses information on the underlying dynamics, and so the phases have to be added in by hand.

Since all generating functions $h(x_0, x_1)$ are defined only up to the addition of constants, the correct WKB quantization condition (incorporating the reflection and caustic) can be found by adding the classically zero amount, $-3\pi\hbar/2$, to $h(x_0, x_1)$ when it is first defined in (21). This leaves the generating function identical to the full action *including* the correct 'Maslov' phase expected for the paths under consideration. These are paths with fixed angular momentum which start at all points on the boundary, and hence travel as a family with a caustic half way along their length.

Unfortunately, it is not clear how the Maslov phase should be included for billiards (or other systems) where the families of trajectories form caustics in certain regions of coordinate space, but not others. The 2-torus studied above was a special case since, with no reflections or caustics, the phases are all zero. For completely ergodic billiards, the lack of any conserved quantity like angular momentum might stop the caustics appearing, but the orbits will still all undergo one reflection and so the generating function should at least have $-\pi\hbar$ added on. For the boundary maps from integrable billiards, then if the conserved quantity can be identified as the momentum in the problem, it should be possible to apply the same argument as above. Otherwise, it is not clear from this how any extra phases should appear.

Perhaps the most important point however, is that (38) will only give the *semiclassical* eigenvalues at best. Yet the quantization was done exactly and at no point was the semiclassical limit $\hbar \to 0$ taken. This focuses attention back onto the meaning of the classical $H_{\rm Mos}$, which has no known analogue. The point is that while $H_{\rm Mos}$ is undoubtedly the correct classical Hamiltonian for the situation it is describing, it corresponds to a fundamentally different classical system to the billiard system it derives from.

This difference comes about because of Moser's specific choice of extrema. In principle, any choice of extremal path would have sufficed provided that (5) could be inverted. But by restricting the motion to being on the PSS, Moser is choosing a different classical flow to that inside the billiard. The crucial point is that all these different classical flows give the same action between any two points along their extrema.

In particular, given any two points on the boundary, the value of the classical action for any extremal motion between the points is, by construction, the same whether calculated using the inside of the billiard, or using the 'action' in the Moser formulation. In order for this to occur it is necessary for the action in Moser's approach to be no longer additive, in the sense that if a given path (on the PSS) is split into segments then the sum of the actions on these segments is not the same as the action of the whole path. Also, as was seen above, the Moser approach is unable to account for the Maslov phases that appear, but provided that these can be added in by hand then the classical information in both is the same for all the extremal paths.

Now consider defining the Feynman path integral to calculate the propagator between two points. It is well known that in the semiclassical limit, the dominant terms are given by the extremal classical paths, with the important quantities appearing being the actions and phases along these paths. In this sense, albeit heuristic, it might be expected that the propagator calculated within Moser's picture would agree with that calculated inside the billiard for a single chord, but only at leading-order. The terms at $O(\hbar^2)$ will be different because the neighbouring non-classical paths that the quantum mechanics will explore will be different.

Nevertheless, if this is true, then one can only expect Moser's approach when applied to a billiard system to produce answers correct to $O(\hbar^2)$ which is exactly what has been found above. For the circle, the difference from the exact eigenvalues is only about 6% of the mean-level spacing which is itself at $O(\hbar^2)$ (see [6] for more details). Indeed, the Moser result is *only* the leading-order result in this case. It is not known whether this would be the case for all billiards. Certainly, the average error will not grow in the mean-level spacing if the above argument holds, but it could be sufficiently large to obscure the exact spectrum.

Achieving accuracy to $O(\hbar^2)$ is also reminiscent of the errors incurred by making a canonical transformation, [26]. Now, Moser's approach is clearly not equivalent (in this case) to simply canonically transforming the coordinates, since it results in a change in both the dimension of phase space and in the classical dynamics. However, it does bear some of the hallmarks of quantizing using action-angle variables. This does involve a canonical transformation, and when an integrable system is viewed using action-angle variables, the caustics are not evident. Thus if the system is quantized in this representation, the quantum results will differ in two ways from those obtained when quantizing in a more typical coordinate representation. First, the canonical transformation will result in differences at $O(\hbar^2)$, and secondly the Maslov phases associated with the caustics will not appear. Both of these problems also arise in Moser's technique, though at present the precise connection with the action-angle approach is not understood.

5. Iterating the mapping

Another question that can be addressed in this approach is whether the iteration of the map commutes with quantization of the Moser Hamiltonian. This can be tested by, for example, showing that the generating function for N mappings, $h(x_0, x_N)$, leads to quasi-energies $\varepsilon_m^{(N)}$ such that $\varepsilon_m^{(N)} = N\varepsilon_m^{(1)}$. However, not all monotonic twist maps remain monotonic twist maps under iteration, as Moser points out in [1]. The twist condition (1) means that a small vertical line element, $(0, \delta p_0)$, will be mapped to a new element, $(\delta x_1, \delta p_1)$ lying at an angle θ to the vertical where $0 < \theta < \pi$ taken in the clockwise direction. Iterating such a map once will cause the line element to end up at an angle 2θ to the vertical which can cover the range $0 < 2\theta < 2\pi$ and is thus not necessarily a monotonic twist mapping.

The new monotonic twist condition is clearly $\partial x_2/\partial p_0 > 0$. For the generalized standard map it is easy to show that

$$\frac{\partial x_2}{\partial p_0} = 2 - V''(x_1) \tag{39}$$

where the primes indicate differentiation with respect to position. For the Chirikov–Taylor mapping, $V(x) = K\cos(2\pi x)$, this is monotonic if $K < 1/2\pi^2$, but no way was found for constructing the exact Moser Lagrangian for the once-iterated map. Since the allowed values of K are very small, however, this suggests a perturbative approach at first order in K, or more generally $|V|$. For general $V(x)$ this results in the following approximate generating function:

$$h(x_0, x_2) = \frac{(x_2 - x_0)^2}{4} - V\left(\frac{x_2 + x_0}{2}\right) - V(x_2) \tag{40}$$

which is like using the 'weak perturbation' approximation, $x_1 = (x_2 + x_0)/2$. This in turn leads to the Moser Lagrangian

$$L_{\text{Mos}}(x, \dot{x}, t) = \frac{\dot{x}^2}{4} + \frac{1}{4(t-1/2)^2}\left[V(x - \dot{x}(t-1/2)) - V(x) + \dot{x}(t-1/2)V'(x)\right] - \dot{x}V'(2t - 1/2) - 2V(x) \tag{41}$$

which gives $\ddot{x} = 0$, and integrates to exactly give (40) for all potentials. In spite of appearance, the Lagrangian is perfectly well behaved at $t = 1/2$ as can easily be checked. However, in converting to the Moser Hamiltonian, momentum- and time-dependent potentials appear that made quantization impossible. Thus, it was not possible to check whether iteration and quantization commute for this case, although more work is being done on this question.

For the examples of the 2-torus and circle billiard however, the check can be made very easily. In both cases, the momentum is unaffected by the iteration while for N mappings (or $N - 1$ iterations) the positions vary as follows:

$$\text{2-torus} \qquad x_N = x_0 + \frac{Nap_0}{\sqrt{P^2 - p_0^2}} \tag{42}$$

$$\text{Circle billiard} \qquad x_N = x_0 + \frac{N}{\pi}\cos^{-1}\left(\frac{p_0}{2\pi Pa}\right). \tag{43}$$

In both cases this leads to classical Moser Hamiltonians corresponding to N mappings, $H_{\text{Mos}}^{(N)}$, being given by $H_{\text{Mos}}^{(N)} = N \times H_{\text{Mos}}^{(1)}$. Hence, the quasi-energies scale in exactly the way that they do when the quantum unitary operator is iterated: $U^{(N)} = U^N$. This proves the commutation of the iteration with the quantization for these systems.

6. Discussion

While Moser's idea gives an interesting new slant on the classical and quantum mechanics of area-preserving maps, it is by construction a rather artificial approach. It should therefore be viewed more as a new mathematical technique for studying such mappings rather than a key to understanding the existing physics in a deeper way. The examples given herein are thus intended to show how the Moser technique might be applied, rather than exhaustively study the physics behind the systems.

The main obstacle to studying more generic systems is that, in such cases, the Moser Lagrangian is generally dependent on the fictitious time, and has a highly complex dependency on the position and velocity on the PSS. This makes the Legendre transform to the Hamiltonian, and the ensuing quantization with complicated operator-ordering problems, very difficult; more difficult than was felt necessary for the present needs in view of the subtle points raised above.

Indeed, the most interesting points raised by the quantization of monotonic twist maps using the Moser approach, are just those subtle ones that were outlined above. The classical and quantum properties of Moser's technique when generalized to different motions are not as yet understood. The cat map, for example is known to have an underlying *exponential* flow [13] in a manner very similar to that introduced by Moser. However, it is not known how to invert (5) for this motion. It is probable that operator ordering ambiguities will also appear when different extremal motions are imposed in order to find the Lagrangian in terms of the map. While it is hoped that symmetrizing the operators will consistently deal with these, there is no proof that this is the case. Furthermore, there remain many subtle points surrounding the quantization of families of classical systems all of which have the same action but different dynamics. Study of such questions will be reserved for a later paper.

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